Dinensional analysis: another example.

Heat conductivity

If the temperetaure in a medium is not uniform (somwhere is hotter, somewhere is cloder), then heat flows from hotter places to colder places.

Conduction: matter itself doesn't move

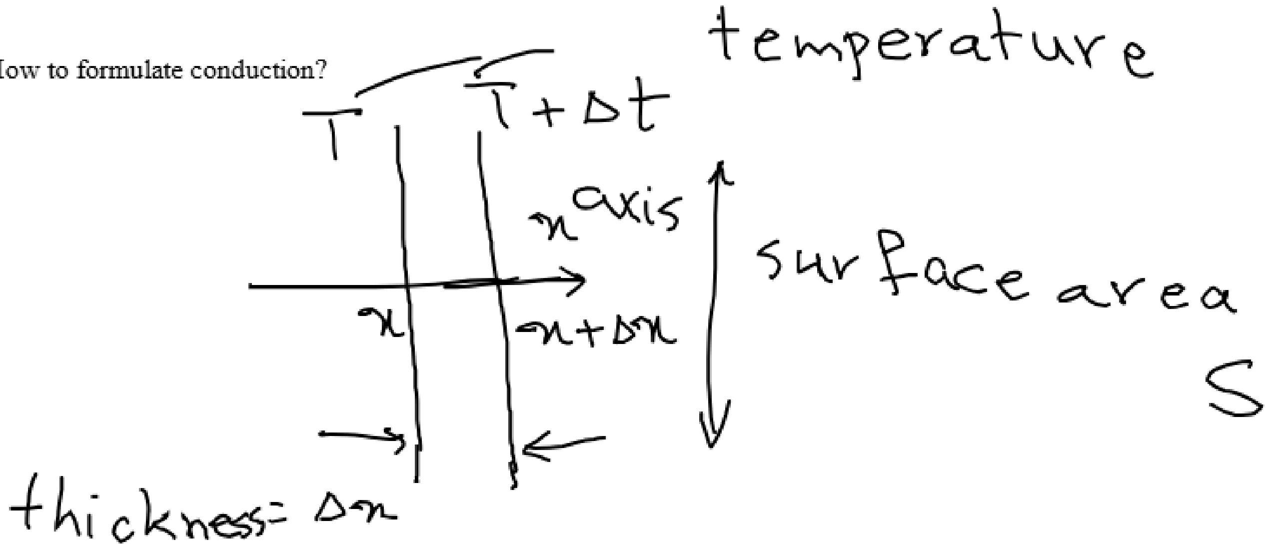
Another way of heat transfer is convection.

Convection: matter itself moves (hotter material moves to colder places, and vice versa).
Example: a heater on the floor. Hot air goes up, cold air goes down

Example for conduction: a metal rod is heated from one end. Heat moves from the heated end to the other end. The matter itself doesn't move.

Another way of heat transfer is radiation.

How to formulate conduction?



The temperature difference is ΔT

$$\frac{\text{heat transfer}}{\text{time}} = \frac{\text{energy transfer}}{\text{time}}$$

$$\frac{\text{energy transfer}}{\text{time}} = f(\Delta T)$$

$$\Delta T = 0 \implies f(\Delta T) = 0$$

A Taylor expansion of f for small ΔT

$$f(\Delta T) = f(0) + [f'(0)]\Delta T + \cdots$$

$$f(0) = 0 \qquad f(\Delta T) \propto (\Delta T)$$

$$\frac{\text{energy transfer}}{\text{time}} \propto (\Delta T)$$

$\langle \Delta T \rangle$ depends on $\langle \Delta x \rangle$

something proportional to $\langle \Delta T \rangle$,

which remains meaningful

at $\langle \Delta x \rangle \to 0$

This thing is the derivative

(I'm using an argument similar

to the one used for viscosity)

the rate of heat transfer =

$$\frac{\text{heat transfer}}{\text{time}} \propto \frac{\Delta T}{\Delta x} \longrightarrow \frac{dT}{dx}$$

$S \longrightarrow \alpha S$

the rate of heat transer $\longrightarrow$

α times the initial rate

the rate of heat transfer =

$\underbrace{\text{the heat current}}_{I_Q}$ = (energy transfer per time)

$\propto$ the surface area S

$I_Q \propto S$

$I_Q \propto \frac{dT}{dx}$

$\Rightarrow I_Q \propto S \frac{dT}{dx}$

To change this proportionality relation into an equality relation, a constant is needed:

$$I_Q = -KS\frac{dT}{dx}$$

heat conductivity

the reason for the minus sign:

If T is increasing x (the right side of the layer is hotter that the left side of the layer), then heat transfer is from the right to the left, and using a conventionalpositive sign for the heat transfer to the right, the heat current I_Q should be negative.
The same is true for the case that T is decreasing in x, so that the derivative of x is negative. This time I_Q should be positive,and again the negative sign at the right-hand side of the equation is needed.

The dimension of K can be deduced, using that equation:

$$[I_Q] = \left[\frac{\text{energy}}{\text{time}}\right] = \frac{ML^2T^{-2}}{T} = ML^2T^{-3}$$

I_Q → power

T → dimension of time

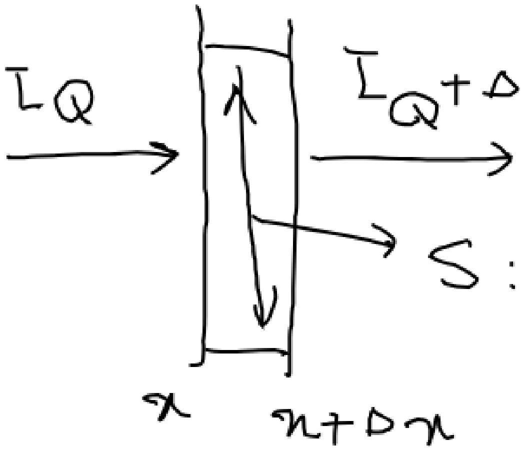
$[S] = L^2$ [The derivative of temperature] $= \Theta L^{-1}$

Θ → dimension of temperature

$$ML^2T^{-3} = [K]\, L^2\, \Theta L^{-1}$$

$$[K] = \frac{MLT^{-3}}{\Theta}$$

The SI unit of $K = \dfrac{W}{K\,m}$



S: the surface area of the layer

The net amount of energy gained by the layer during the time $\Delta t = [I_Q - (I_Q + \Delta I_Q)]\,\Delta t$

The net amaunt of energy

gained by the layer $= -(\Delta \Sigma_Q) \Delta t$

This causes a change in the temperature of the layer

$= C \; (\Delta T)$ temperature change

the heat capacitiy of the layer

One arrives at

$$C \, \Delta T = -(\Delta I_Q)\, \Delta t$$

$C = c\,(\Delta V)$ the volume of the layer

$\Delta V = S\,\Delta x$

c: $\dfrac{\text{heat capacity}}{\text{volume}}$

So: $cS(\Delta x)\,\Delta T = -(\Delta I_Q)\,\Delta t$

$$\Rightarrow \quad cS\frac{\Delta T}{\Delta t} = -\frac{\Delta I_Q}{\Delta x}$$

The temperature and the heat current, both are function of t (time) and x (position).
So the notation of partial derivative is used:

$$f(t,x) \qquad \frac{\partial f}{\partial t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t, x) - f(t,x)}{\Delta t}$$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(t, x+\Delta x) - f(t,x)}{\Delta x}$$

$$cS\frac{\partial T}{\partial t} = -\frac{\partial I_Q}{\partial x}$$ the continuty equation

$$I_Q = -KS\frac{\partial T}{\partial x}$$ the heat conductivity equation

Combining the two equations

(eliminating I_Q),

$$c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right)$$

Consider the case that c and K are constants.

$$\frac{\partial T}{\partial t}=D\frac{\partial^2 T}{\partial x^2} \quad D=\frac{K}{c} \text{ a constant}$$

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$ This is a partial differential equation for T

Newton's second law

$$m \frac{d^2 x}{dt^2} = F\left(t, x, \frac{dx}{dt}\right)$$

An ordinary differential eqution.

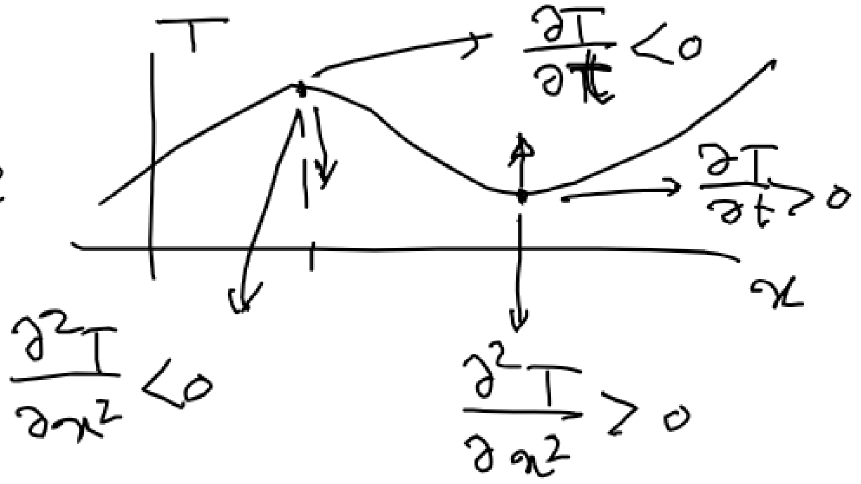
(It doesn't contain partial derivative)

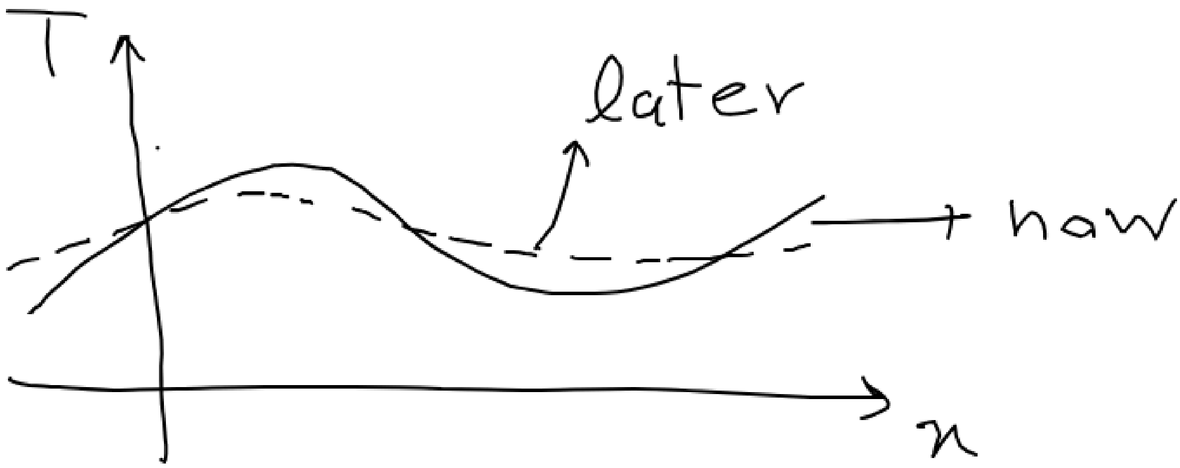
$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} \qquad D = \frac{K}{c}$$

A diffusion equation

D : the diffusion constant

At some time





The maximum goes down
The minimum goes up

The temperature becomes more uniform.

The diffusion equation makes things more uniform.

The dimensional analysis:

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$ temperature

$$\Theta T^{-1} = [D]\, \Theta L^{-2}$$

$$[D] = L^2 T^{-1}$$ → the dimension of time

What is the time t needed for two points separated by a distance x to have almost the same temperatures?

$1 \xleftrightarrow{x} 2$

initially $T_2 \neq T_1$

after a time t, $T_2 \approx T_1$

What is the relation between t and x.

Dimensional analysis

t, x, D ⟵ the relevant parameters

One has to find a dimensionless combination of these:

$[t^\alpha x^\beta D^\gamma] = 1$

$T^\alpha L^\beta (L^2 T^{-1})^\gamma = 1$

$T: \ \alpha - \gamma = 0$

$L: \ \beta + 2\gamma = 0$

$\gamma = \alpha$ $\qquad \left(\frac{Dt}{x^2}\right)^{\alpha}$

$\beta = -2\alpha$

Only one independent dimensionless combination

$$\frac{Dt}{x^2} = \text{constant}$$

That is $\qquad t = \frac{(\text{constant})\, x^2}{D}$

$t \propto x^2$

Multiplying the size by α (say 2), the time is multiplied by α^2 (say 4).

An example in cooking:

It takes about 10 minutes (maybe a little less) to cook an egg (so that it is hard boiled) egg of chicken

How much time is needed to cook an ostrich egg?

Ostrich ?

A huge bird, which doesn't fly.

The persian name for it:

Camel-chicken

The mass of an ostrich egg
is about 1.5 kg
1 → 2 kg
The mass of ordinary egg
is about 50 g.
So the mass ratio is about
30

The volume ratio ≈ 30

The densities are the same.

$= (\text{The size ratio})^3$

The size ratio ≈ 3 $\quad 3^3 = 27$

The time ratio $= (\text{the size ratio})^2$

The time ratio $\approx 3^2 \approx 10$ $\quad 3^2 = 9$

$$\text{time}_{\text{ostrich}} \approx 10 \underbrace{\text{time}_{\text{chicken}}}_{\text{10 minutes}}$$

$$\text{time}_{\text{ostrich}} = 100 \text{ minutes}$$

$$\approx 1.5 \text{ hours}$$