Scaling. Scaling relations between quantities.

Let's begin with two quantities X and Y.

X and Y are related to each other. Y is a function of X.

$Y = f(X)$

This relation can be dedimensionalized (its dimensions be removed) with two quantities $x0$ and $y0$, where $x0$ has the dimension of X and $y0$ has the dimension of Y:

$[x0] = [X]$ $[y0] =[Y]$

How is this achieved?

$Y = y0\ g(X/x0)$ $[Y/y0] = 1$ $[X/x0] = 1$ $(Y/y0) = g(X/x0)$

$f(X) =: y0\ g(X/x0)$ $g(s) = (1/y0)\ f(x0\ s).$

$(X/x0)$ and $(Y/y0)$ are dimensionless.

I am asking this question.

What happens to Y, if X is multiplied by a?

I mean, if $y1 = f(x1)$, $x2 = a\,x1$, $y2 = f(x2)$

What is the relation between y2 and y1?

The ratio (y2/y1) is like this $(y2/y1) = [g(x2/x0)]/[g(x1/x0)]$ <-- This relation contains x0

$f(X) = y0\, g(X/x0)$

What is the condition that the ratio (y2/y1) is independent of x0?

That is, what is the condition that x0 (which has the dimension of X), has no effect on the ratio (y2/y1)?

Suppose that this is the case, that (y2/y1) is independent of x0.

So x0 can be put equal to anything, and the result for (y2/y1) won't change.

Among these arbitrary values for x0, one can choose a specific one: $x0 = x1$.

The result is $(y2/y1) = [g(x2/x1)]/[g(1)] =: h(x2/x1)$

So far, it has been proved that $(y2/y1) = h(x2/x1)$

(Of course based on teh assumption that x0 doesn't appear in the ratio $(y2/y1)$.

$(y3/y2) = h(x3/x2)$, $(y3/y1) = h(x3/x1)$, $(y2/y1) = h(x2/x1)$.

Obviously, $(y3/y1) = (y3/y2)\,(y2/y1)$

This relation between the left-hand sides, results in a similar relation between the right-hand sides:

$h(x3/x1) = h(x3/x2)\,h(x2/x1)$

This is a functional relation for h.

Of course, $(x3/x1) = (x3/x2)\,(x2/x1)$

Denote $(x3/x2)$ and $(x2/x1)$ by u and v, respectively.

The functional relation for h becomes

$h(v\,u) = h(v)\,h(u)$

What is the form of h, if h is supposed to satisfy this relation?

h(v u) = h(v) h(u)

To solve this relation,

The above holds for all v and u, including the case v = u.

So, h(u u) = h(u) h(u), that is, h(u^2) = [h(u)]^2

h of the square of u, is equal to the square of h(u).

This argument can be repeated: h(u u^2) = h(u) h(u^2) = h(u) h(u) h(u)

So, h(u^3) = [h(u)]^3

Continuing in this way, onr arrives at

h(u^n) = [h(u)]^n, for any positive integer n.

integer: 0, +1, -1, +2, -2, ...

positive integer (natural number): 1, 2, 3, ...

If you wish, you can be more rigorous and prove that by induction

That is,

Prove the statement for n =1:

$h(u^1) = [h(u)]^1$

This is obviously correct, as it is $h(u) = h(u)$

Next, assume that the statement is correct for $n = m$: $h(u^m) = [h(u)]^m$

And prove that the statement is correct for $n = m+1$

$h[u^{(m+1)}] = [h(u)]^{(m+1)}$

This has to be proved.

$h[u^{(m+1)}] = h(u\, u^m)]$

According to the functional relation, $h(u\, u^m) = h(u)\, h(u^m)$

According to the assumtion, that the statement is correct for $n = m$, $h(u^m) = [h(u)]^m$

The result is that $h(u\, u^m) = h(u)\, [h(u)]^m$ But this means $h[u^{(m+1)}] = [h(u)]^{(m+1)}$

So the statement is also correct for $n = m+1$.

This concludes the proof ofthe staement (for all positive integers) by induction.

Now we know that if n is a positive integer, then $h(u^n) = [h(u)]^n$.

This is readily generalized to all integers:

First,

$h(u) = h(1\ u) = h(1)\ h(u)$

This results in $h(1) = 1$

This proves the statement for $n = 0$: $h(u^0) = [h(u)]^0$

Because both sides are 1.

What about negative integers? Suppose that n is a negative integer. Then (-n) is a positive integer.

So, $h[u^{(-n)}] = [h(u)]^{(-n)}$

But, as a result of the functional relation for h, $h(1) = h[u^n\ u^{(-n)}] = h(u^n)\ h[u^{(-n)}]$

As $h(1) = 1$, this results in $h(u^n) = 1/\ \{h[u^{(-n)}]\} = 1/\ \{[h(u)]^{(-n)}\} = [h(u)]^n$

So the statement is correcet, even if n is a negative integer.

We have proved that the statement is correcet for all integers: positive, zero, negative.

h(u^n) = [h(u)]^n,   for all integers n

Going further, suppose that n is a nonzero integer.

So (1/n) is well-defined.

Question: Is the statement also correct if n is substitued by (1/n)?

That is, is it correct that   h[u^(1/n)] = [h(u)]^(1/n)?

To check this, one way is to examine h(u)

u = [u^(1/n)]^n,
So, h(u) = h{[u^(1/n)]^n} = {h[u^(1/n)]}^n

Taking the n'th root of the above,   [h(u)]^(1/n) = h[u^(1/n)].

Taking the n'th root of something, meansraising that thing to the power (1/n)

So the statement is correct if n is substituted with (1/n).

Let's see what we have up to now:

If m is an integer, $h(u^m) = [h(u)]^m$

If n is a nonzero integer, $h[u^{(1/n)}] = [h(u)]^{(1/n)}$

Let's combine these two:

$$h[u^{(m/n)}] = h\{[u^{(1/n)}]^m\} = \{h[u^{(1/n)}]\}^m = \{[h(u)]^{(1/n)}\}^m = [h(u)]^{(m/n)}$$

So, if m and n are integers, and n is not zero, then
$h[u^{(m/n)}] = [h(u)]^{(m/n)}$

Any rational number q is equal to a ratio (m/n), where m and n are integers and n is not zero.

This means that for all rational numbers q,
$h(u^q) = [h(u)]^q$

I am going to extend this to real numbers.

Any real number r, is the limit of a sequence q of sational numbers:

$r = \text{limit (j to infinity) } q_j$

q_j can be taken as the decimal representation of r, truncated at j numbers after the decimalpoint.

Example:

r is the square root of 2:
r^2 = 2

r is real, but not rational.

r = 1.41411356 ...

the decimal representation of r

q_1 = 1.4, q_2 = 1.41, q_3 = 1.414, ...

h(u^r) = h{u^[(limit-(j to infinity)q_j]} = h{limit_(j to infinity)[u^(q_j)]}

(Exponentiation is continuous)

Assume that h is continuous. Then,

h{limit_(j to infinity)[u^(q_j)]} = limit_(j to infinity){h[u^q_j)]} = limit_(j to infinity) {[h(u)]^(q_j)}
= [h(u)]^[limit_(j to infinity) q_j] = [h(u)]^r

The final result: if r is real, then
h(u^r) = [h(u)]^r.

h is a power function.

$h(u) = ?$

$u = \exp(w)$, which means that $w = \ln u$.

$h(u) = h(e^w) = [h(e)]^w = e^{w \ln[h(e)]} = u^{\ln[h(e)]}$

$h(u) = u^c$, where $c = \ln[h(e)]$ is a constant

What about f?
It was seen that $f(x) = y0\ g(x/x0) = y0\ g(1)\ h(x/x0)$

So,
$f(x) = y0\ g(1)\ (x/x0)^c$

$f(x) = b\ x^c$

This is the final result.

If the relation between Y and X doesn't depend on scales, scales are canceled in the relation between the ratios (y2/y1) and (x2/x1), then the only possible relation between Y and X is a power law:

$Y = b\ X^c$, where b and c are constants.

Of course it was assumed that the relation between Y and X is continuous.

Examples:

The surface area S and the side L of a square: $S = L^2$

The volume V and the side L of a cube: $V = L^3$

The mass M and the radius R of a ball: $M = b R^3$

b is equal to $(4 \pi/3)$ times the density of the ball.