Let'us discuss a very simple model for movement.

By this, I mean something like this.

Somebody (or something) starts moving, say walking, running,or swimming.

The motion begins from rest.

That is, at the time zero, the velocity is zero.

The displacement (relative to the time zero) is denoted by x.

The motion is assumed to be in one dimension (to make things simpler).

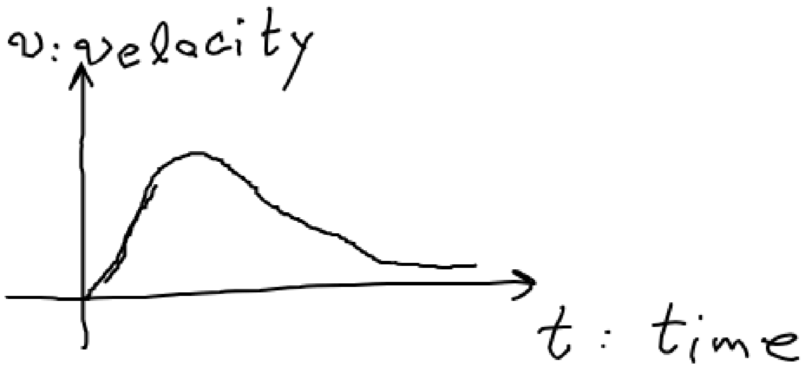
Some observations:

The velocity increases with time (at first).

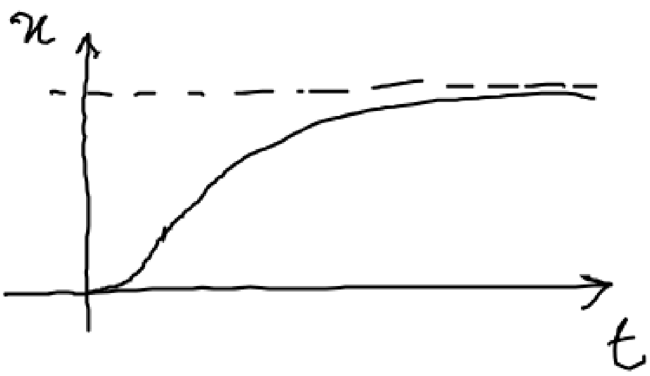
But after a while, the mover gets tired, and the velocity begins to decrease.

At some point, it could happen that the mover stops moving.

A qualitative plot for the velocity is the following.



A qualitative plot for the displacement x is the following



This is for the case that at t tending to infinity, x tends to a finite value.

The aim is to build a simple mathematical expression for the displacement, which shows these features.

First, begin with the displacement at small times.

One can use a Taylor expression for x as a function of time t:

A Taylor expansion:

$x(t) = x(0) + [x'(0)]\, t + [x''(0)]\, (t^2/2) + \ldots$

prime means differentiation.

$x'(0)$ is the first derivative of x at $t=0$.
$x''(0)$ is the second derivative of x at $t=0$.

The first derivative of the displacement is the velocity.
The second derivative of the displacement is the acceleration.

So the equation is like this

x(t) = x(0) + [v(0)] t + [a(0)] (t^2/2) + ...

The acceleration is denoted by a.

As the mover begins from rest, the velocity at t=0 is zero.

v(0)=0.

x is the displacement from t=0, so it is equal to zero at t=0.

x(0)=0.

The result is

x(t) = [a(0)] (t^2/2) + ...

$$x(t) \approxeq \frac{a(0)\, t^2}{2},$$

A is approximately equal to B

$A \approxeq B$

for small times.

The exact relation would be

$$x(t) = \frac{a(0)\, t^2}{2} f(t)$$

This is the definition of f:

$$f(t) := \frac{x(t)}{\frac{a(0)\, t^2}{2}} = \frac{2x(t)}{a(0)\, t^2}$$

$$x(t) = \frac{a(0)\, t^2}{2} f(t)$$

What is the form of f?

We already had

$$x(t) \simeq \frac{a(0)\, t^2}{2}, \quad \text{for small } t$$

To arrive at this, the small-t behavior of f should be

$$f(t) \simeq 1, \quad \text{for small } t$$

Assuming that $x(t)$ tends to a constant as t tends to infinity,

$$\lim_{t \to \infty} x(t) = L,$$

that is $$\lim_{t \to \infty} \frac{a(0)\, t^2}{2} f(t) = L$$

Or, $$\frac{a(0)\, t^2}{2} f(t) \approx L, \text{ for large } t$$

divide this by $\left[\frac{a(0)t^2}{2}\right]$:

The result is

$$f(t) \approx \frac{L}{\frac{a(0)t^2}{2}} = \frac{2L}{a(0)t^2}, \quad \text{for large } t$$

That is, $f(t) \approx \frac{\tau^2}{t^2}$, for large t

$$\tau := \sqrt{\frac{2L}{a(0)}}$$

f is, of course, dimensionless,

because $[x(t)] = \left[\frac{a(0)t^2}{2}\right]$.

As f is dimensionless,

the dimension of τ is the same as the dimension of t.

The dimension of τ is time

So τ is a specific (characteristic) time.

$$f(t) \approx \begin{cases} 1, & \text{for small } t \\ \dfrac{\tau^2}{t^2}, & \text{for large } t \end{cases}$$

Or,

$$\frac{1}{f(t)} \approx \begin{cases} 1, & \text{for small } t \\ \dfrac{t^2}{\tau^2}, & \text{for large } t \end{cases}$$

A very simple form for $\frac{1}{f}$, which has those features, is

$$\frac{1}{f(t)} = 1 + \frac{t^2}{\tau^2}$$

This is not the only form. This is just a simple example

Perhaps this is the most simple form for (1/f), which has those features.

Let'u work with this form.

$$f(t) = \frac{1}{1 + \left(\frac{t}{\tau}\right)^2}$$

$$x(t) = \frac{[a(0)/2]\, t^2}{1 + \left(\frac{t}{\tau}\right)^2}$$

This can be rewritten as

$$x(t) = \frac{[a(0)\, \tau^2/2]\, (t/\tau)^2}{1 + (t/\tau)^2}$$

simplifying this form,

$$\left[\frac{t}{\tau}\right] = 1 \qquad [\tau] = \text{time}$$

$$\left[\frac{a(0)\tau^2}{2}\right] = \text{length}$$

$\dfrac{a(0)\tau^2}{2}$ is a length scale.

Denote it by ℓ

$$[\ell] = \text{length}$$

And the expression for x becomes

$$x(t) = \ell \frac{\left(\frac{t}{\tau}\right)^2}{1 + \left(\frac{t}{\tau}\right)^2}$$

This can be written as

$$\frac{x}{\ell} = \frac{(t/\tau)^2}{1 + (t/\tau)^2}$$

Now, everything has been expressed in terms of dimensionless quantities: $\frac{x}{\ell}, \frac{t}{\tau}$

In this simple model, there are only two constants: l (a length scale), and τ (a time scale).

Let's discuss the time dependence of v (the instantanaus velocity) and $\bar{v}$ (the average velocity)

For v: $\quad v = \dfrac{dx}{dt} \qquad x = \ell \dfrac{\left(\frac{t}{\tau}\right)^2}{1+\left(\frac{t}{\tau}\right)^2}$

$$v = \frac{\ell}{\tau}\, 2 \frac{\frac{t}{\tau}}{\left[1+\left(\frac{t}{\tau}\right)^2\right]^2}$$

For $\bar{v}$: $\quad \bar{v} = \dfrac{x}{t} \qquad \bar{v} = \dfrac{\ell}{\tau} \dfrac{\frac{t}{\tau}}{1+\left(\frac{t}{\tau}\right)^2}$

There is a time t_s at which v is maximum, and there is a time t_m at which $\bar{v}$ is maximum:

So: $v'(t_s) = 0$

$\bar{v}'(t_m) = 0$

$$v'(t) = \frac{2l}{\tau^2} \left\{ \frac{1 \times \left[1 + \left(\frac{t}{\tau}\right)^2\right]^2 - \left(\frac{t}{\tau}\right) \times 2 \times \frac{2t}{\tau} \times \left[1 + \left(\frac{t}{\tau}\right)^2\right]}{\left[1 + \left(\frac{t}{\tau}\right)^2\right]^4} \right\}$$

$$= \frac{2l}{\tau^2} \frac{1 - 3\left(\frac{t}{\tau}\right)^2}{\left[1 + \left(\frac{t}{\tau}\right)^2\right]^3}$$

$$0 = v'(t_s) \qquad 0 = 1 - 3\left(\frac{t_s}{\tau}\right)^2$$

$$\bar{v}'(t) = \frac{l}{\tau^2} \frac{1 \times \left[1 + \left(\frac{t}{\tau}\right)^2\right] - \frac{t}{\tau} \times 2\frac{t}{\tau}}{\left[1 + \left(\frac{t}{\tau}\right)^2\right]^2}$$

$$= \frac{l}{\tau^2} \frac{1 - \left(\frac{t}{\tau}\right)^2}{\left[1 + \left(\frac{t}{\tau}\right)^2\right]^2}$$

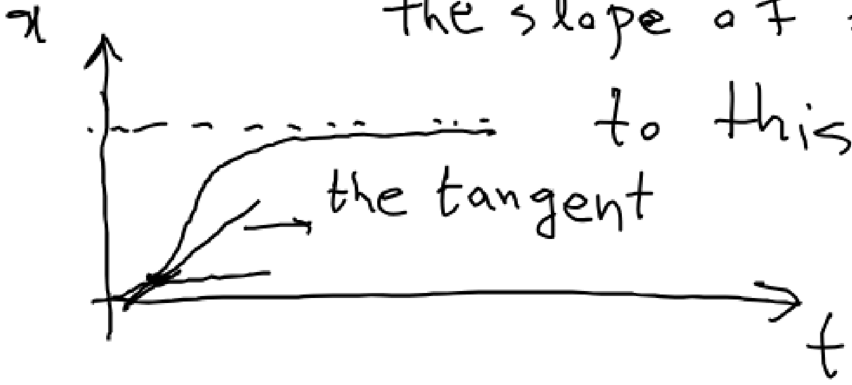
$$0 = \bar{v}'(t_m) \qquad 1 - \left(\frac{t_m}{\tau}\right)^2 = 0$$

$$1 - 3\left(\frac{t_s}{\tau}\right)^2 = 0$$

$$t_m = \tau \quad , \quad t_s = \frac{\tau}{\sqrt{3}}$$

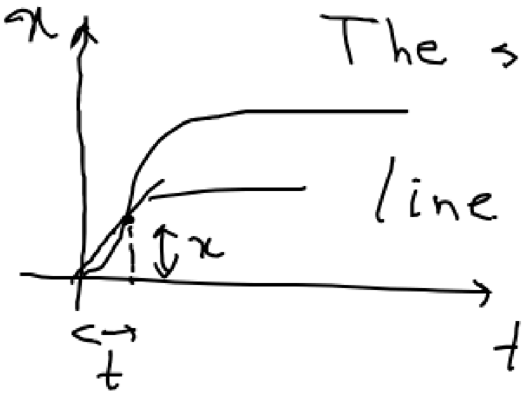
A graphical discussion:

the slope of the tangent to this graph at each time is

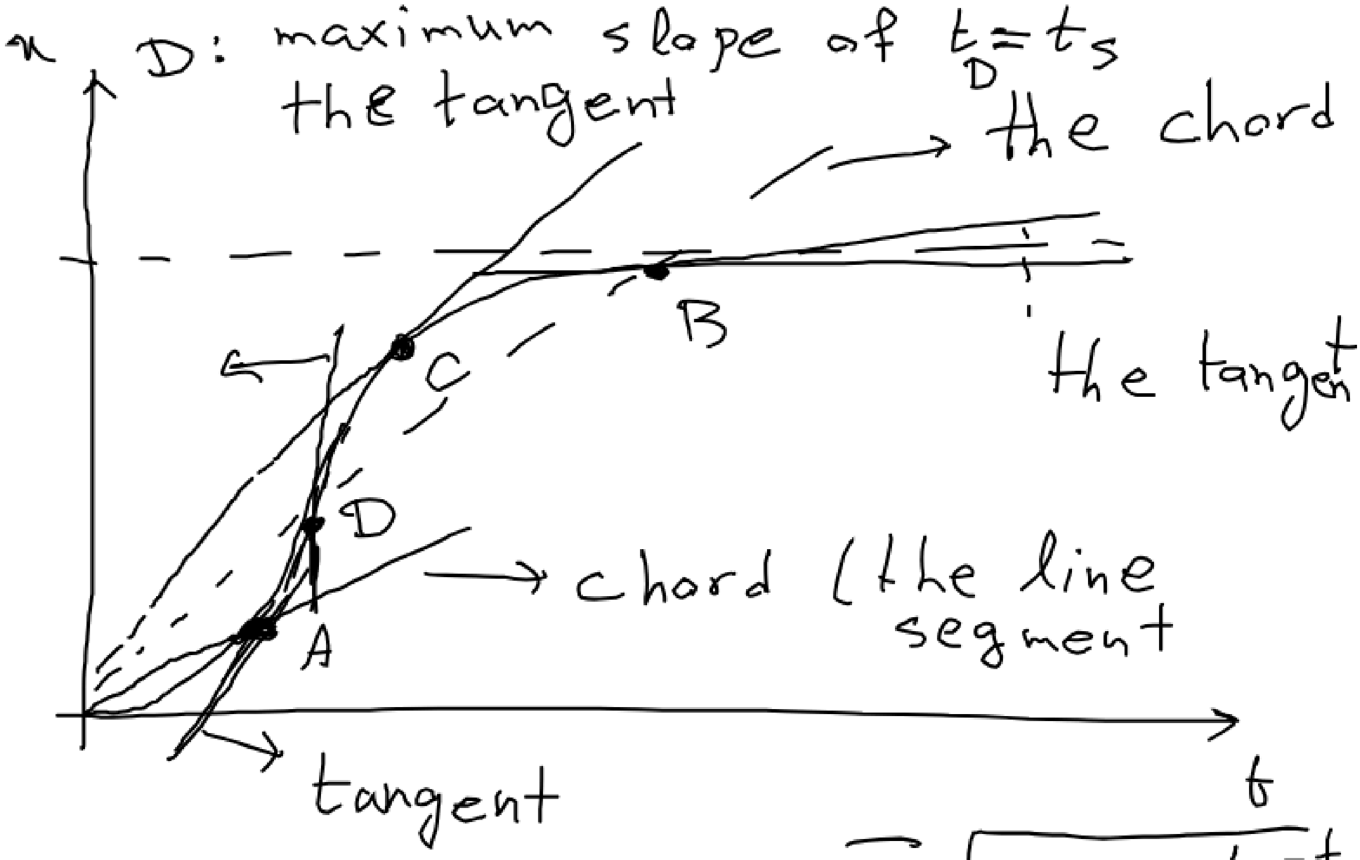


the instantanaus velocity

For the average velocity



The slope of the line segment joining the origin to a point on the curve is the average velocity at that point.



A: $v > \bar{v}$ B: $v < \bar{v}$ $v = \bar{v}$ $t_C = t_m$ tangent

C: The chord is the same as the

It is seen that $t_s < t_m$

(consistent with the result already obtained: $t_s = \frac{\tau}{\sqrt{3}}$ $t_m = \tau$ for that simple model)

Also, t_m is the time $v(t_m) = \bar{v}(t_m)$

This is also consistent with the result of the simple model:

For that simple model,

$$t_m = \tau \qquad v(t_m) = \frac{l}{\tau}\,\frac{2\left(\frac{\tau}{\tau}\right)}{\left[1+\left(\frac{\tau}{\tau}\right)^2\right]^2} = \frac{l}{2\tau}$$

$$\bar{v}(t_m) = \frac{l}{\tau}\,\frac{\frac{\tau}{\tau}}{1+\left(\frac{\tau}{\tau}\right)^2} = \frac{l}{2\tau}$$

$$v(t_m) = \bar{v}(t_m)$$

Qualitatively

$0 \qquad t_s \qquad t_m$

$v > \bar{v}$ $\qquad$ $v < \bar{v}$

$v = \bar{v}$ $\qquad$ $\bar{v}$ is maximum

v is maximum