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Today, let's talk about the law of gravitation and the orbits of planets.

One can begin from the law of gravitation and try to find the orbits,
or begin from the orbits and try to find the law of gravitation.

Newton's law of gravitation is $$\vec{F}_{1\to 2} = -G\frac{m_1 m_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

$\vec{F}_{1\to 2}$ is the force applied on the point mass 2 by the point mass 1.

m1 is the mass of the point mass 1 and m2 is the mass of the point mass 2.

$\vec{r}_1$ is the position of the point mass 1 and $\vec{r}_2$ is the position of the point mass 2.

G is the universal gravitational constant.

As G and m1 and m2 are positive, the force applied on the point mass 2 by the point mass 1 is towards the point mass 1. So this force is an attractive force.

It is also seen that the value of this force is inversely proportional to the distance between the point masses, and also proportional to each of the masses.
G is the proportionality constant.

What about the laws about the orbits?

These are the so called Kepler's laws, originally formulated based on observations.

Historically, Kepler's laws came before the law of gravitation.

The law of gravitation was somehow deduce~~s~~d from Kepler's laws.

The first law:
The orbit of every planet (around the sun) is an ellipse, with the sun being on one of the focal points.

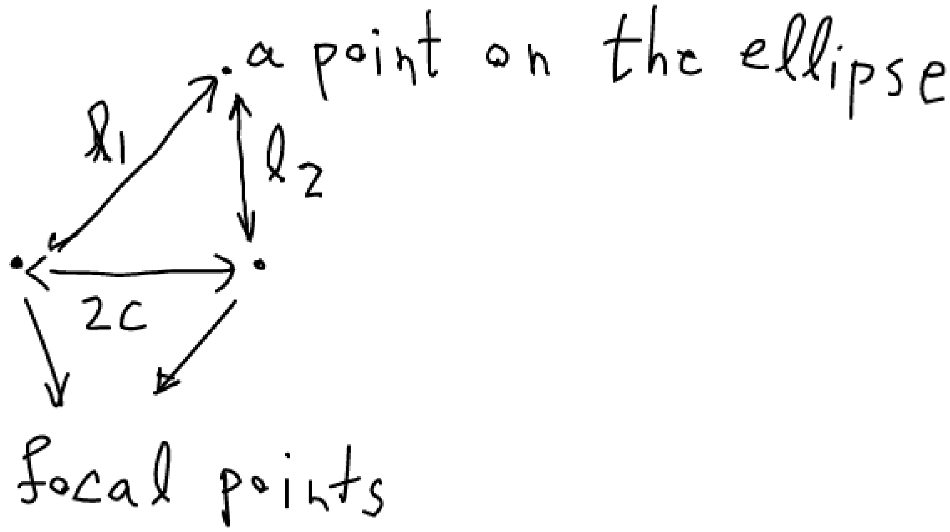
The second law:
The surface area swept by the line joining the sun to the planet, is proportional to the time.

The third law:
The ratio of (square of the orbital period of a planet) to
(the cube of the semi-major axis of its orbit) is a constant
(the same for all planets).

I have to explain the things which have been used to state the laws.

What is an ellipse?

An ellipse is a set of all points on a plane, for which the sum of the distaces from two fixed points (the focal points) is a constant.



The distance between the focal points is denoted by (2c)

$l_1 + l_2 = 2a$ (a constant)

For the triangle with one vertex being a point on the ellipse, and the two other vertices being the focal points, the triangle theorem holds, that the length of each side is not larger than the sum of the lengths of the two other sides:

$$2c \leq l_1 + l_2 = 2a$$

$$c \leq a$$

two limiting cases

1: $c = 0$: The two focal points are the same.

$$l_1 = l_2 \Rightarrow \quad l_1 = l_2 = a$$

So the distance of a point on the ellipse from a single point is a constant (the same for all points on the ellipse). This means that in this case, the ellipse is a circle.

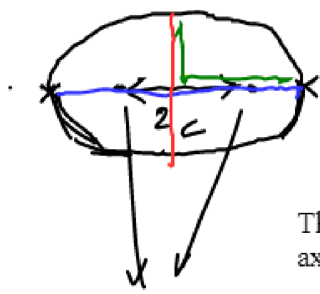
2:

$c = a$

This means that the sum of the lengths of two sides of the triangle is equal to the length of the third side:

$$l_1 + l_2 = 2a = 2c$$

This happens, if and only if the vertex shared between those two sides lies on the third side.

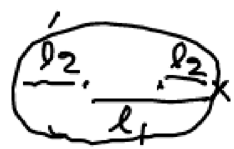


minor axis

major axis

The green line segments are the semi major axis and the semi minor axis: semi major (minor) axis = half the major (minor) axis.

focal points

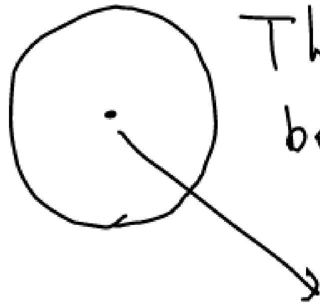


$l_2' = l_2$ because of symmetry

$$2a = l_1 + l_2 = l_1 + l_2' =$$ (the length of) the major axis

So a is (the length of) the semi major axis.
Usually this "the length of" is omitted:
It is simply said that a is the semi major axis.

About the special cases.



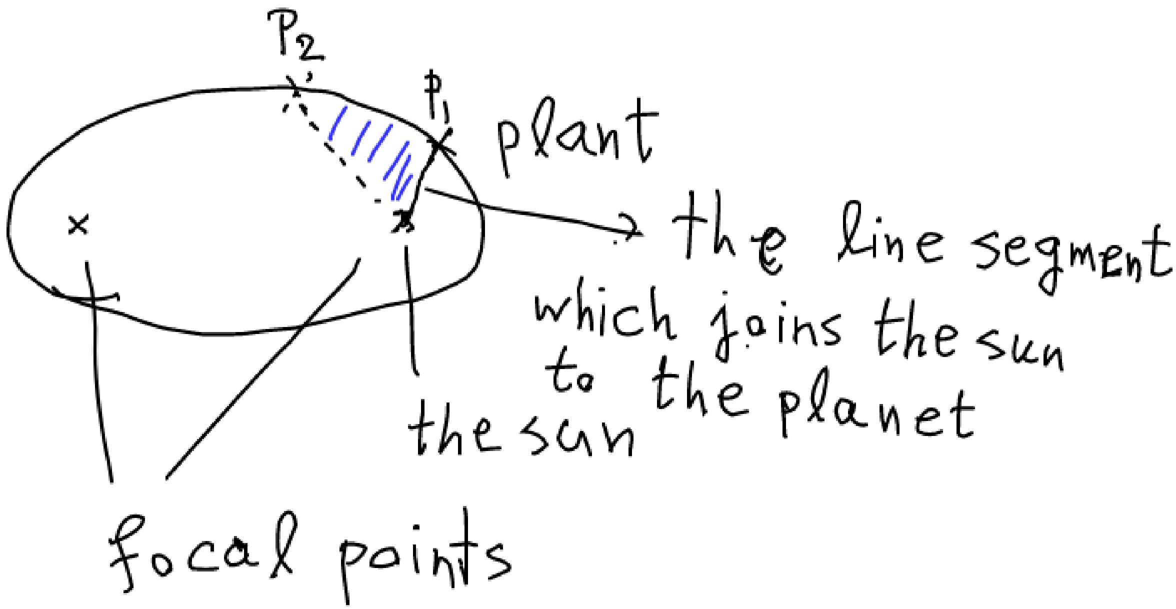
The ellipse becomes a circle.

the two focal points are at the same place

The focal points

$c = a$

The ellipse becomes the line segment which joins the two focal points



As the planet moves, it goes from the point p1 (at the time t1) to the point p2 (at the time t2). The blue region is the surfaces swept by the line joining the sun to the planet. The area of this surface is denoted by (Delta S).

Kepler's second law states that the ratio of this area is proportional to (Delta t), where Delta t = t2 - t1.

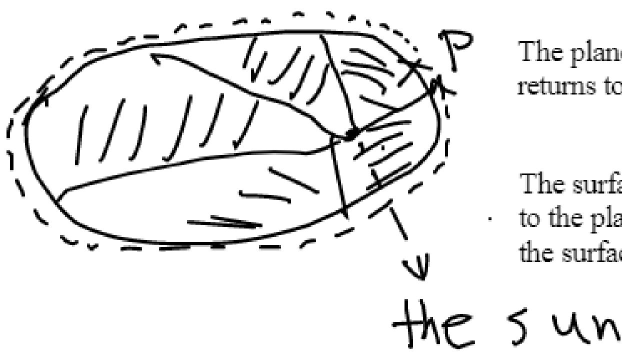
That is to say, [(Delta S) / (Delta t)] is a constant.

This is similar to the case of a uniform motion, where the speed is a constant:
the distance divided by the time is a constant.

But for the motion of planets, the thing which is a constant is the ratio of the surface area to the time: the distance has been replaced by the surface area.

Regarding the third law:
One difference of the third law, with the first and second laws is that the first two laws are about the orbit of a single planet. But the third law is a relation between the orbits of different planets.

The orbital period is the (smallest positive) time that takes for a planet to return to its original position: the time it takes for the planet to repeat its motion.



The planet begins from p, and after the period T returns to p.

The surface area swept by the line joining the sun to the planet during the period T, is of course the surface area of the ellipse.

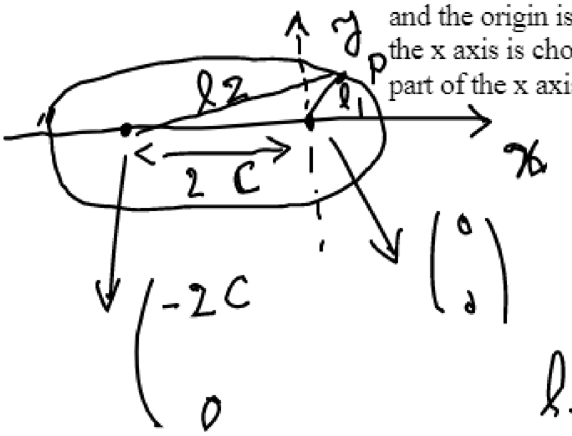
$$\frac{\Delta S}{\Delta t} = \frac{S}{T}$$

For the case that the planet returns to its original position, (Delta S) becomes S, and (Delta t) becomes T.
S is the surface area of the ellipse, and T is the orbital period.

The third law states that (T^2 / a^3) is the same for all planets.
This ratio is independent of the planet (its mass for example),
and independent of the orbit (its shape or position).

The equation for an allipse.

The axes are chosen so that the x axis joins the focal points, and the origin is on one of the focal points. The direction of the x axis is chosen so that other focal point is on the negative part of the x axis.



$$P : \begin{pmatrix} x \\ y \end{pmatrix}$$

$$l_1 = \sqrt{x^2 + y^2}$$

$$l_2 = \sqrt{(x+2c)^2 + y^2}$$

ellipse: $l_1 + l_2 = 2a$

$$\sqrt{x^2+y^2} + \sqrt{(x+2c)^2+y^2} = 2a$$

$$\sqrt{(x+2c)^2+y^2} = 2a - \sqrt{x^2+y^2}$$

squaring the sides

$$\cancel{x^2} + \cancel{4}cx + \cancel{4}c^2 + \cancel{y^2} = \cancel{4}a^2 + \cancel{x^2} + \cancel{y^2} - \cancel{4}a\sqrt{x^2+y^2}$$

$$a^2 - c^2 - cx = a\sqrt{x^2+y^2}$$

again squaring the sides

$$(a^2-c^2)^2 + c^2x^2 - 2c(a^2-c^2)x = a^2(x^2+y^2)$$

$$b := \sqrt{a^2-c^2}$$

$$b^4 = b^2x^2 + 2cb^2x + a^2y^2$$

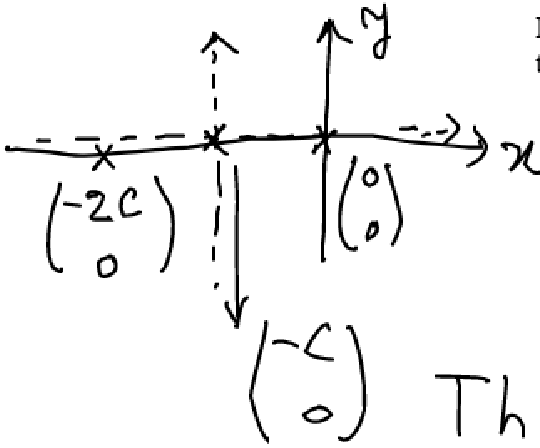
$$b^4 = b^2[(x+c)^2 - c^2] + a^2y^2$$

$$b^2(\underbrace{b^2+c^2}_{a^2}) = b^2(x+c)^2 + a^2y^2$$

divide the sides by a^2b^2

$$1 = \frac{(x+c)^2}{a^2} + \frac{y^2}{b^2}$$

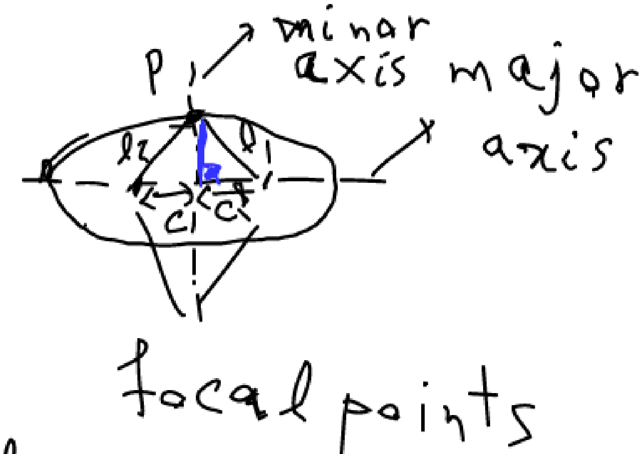
For the dashed axes, the origin is on the middle of the line segment joining the focal points.



Denoting the coordinates corresponding to the dashed axes be primed ones (x', y'), it is seen that
x' = x + c, y' =y

The equation becomes

$$\frac{(x')^2}{a^2}+\frac{(y')^2}{b^2}=1$$



P: $l_2=l_1$

$l_2+l_1=2a \Rightarrow l_2=l_1=a$

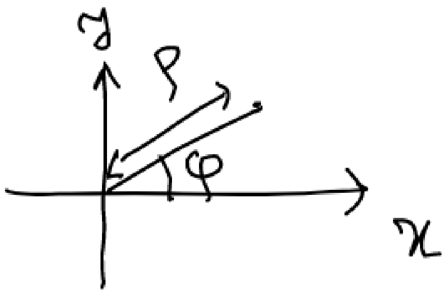
The semi minor axis $=\sqrt{l_1^2-c^2}=\sqrt{a^2-c^2}$

So b is the semi minor axis.

c is half of the focal distance, a is the semi major axis, and b is the semi minor axis.

Finally, let's write the equation of the elleipse in terms of the polar coordinates.

(ρ, φ)



$$x = \rho \cos\varphi \qquad y = \rho \sin\varphi$$

Putting the origin on one of the focal pionts, the x axis on the line joining the focal points, and the other focal point on the negative part of the x axis (these where our first choice for the coordintaes)

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

putting (x,y) in terms of (ρ,φ)

$$\frac{(\rho\cos\varphi + c)^2}{a^2} + \frac{(\rho\sin\varphi)^2}{b^2} = 1$$

$$\left(\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}\right)\rho^2 + \frac{2c\cos\varphi}{a^2}\rho + \overbrace{\frac{c^2}{a^2} - 1}^{-\frac{b^2}{a^2}} = 0$$

$$\rho = \frac{-\frac{c\cos\varphi}{a^2} \pm \sqrt{\left(\frac{c\cos\varphi}{a^2}\right)^2 + \frac{b^2}{a^2}\left(\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}\right)}}{\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}}$$

$$\sqrt{} = \sqrt{\frac{c^2+b^2}{a^4}\cos^2\varphi + \frac{\sin^2\varphi}{a^2}} = \sqrt{\frac{\cos^2\varphi + \sin^2\varphi}{a^2}}$$

So, $$\rho = \frac{-\frac{c\cos\varphi}{a^2} \pm \frac{1}{a}}{\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}}$$

$\rho \geq 0 \quad |c\cos\varphi| \leq c \leq a$

So the solution with the negative sign is not acceptale:

$$\rho = \frac{a - c\cos\varphi}{\cos^2\varphi + \frac{a^2}{b^2}\sin^2\varphi} = \frac{a - c\cos\varphi}{\cos^2\varphi + \frac{a^2}{b^2}(1-\cos^2\varphi)}$$

$$= \frac{a - c\cos\varphi}{a^2 - c^2\cos^2\varphi}\, b^2$$

$$\rho = b^2 \frac{a - c\cos\varphi}{a^2 - c^2\cos\varphi} = \frac{b^2}{a - c\cos\varphi}$$