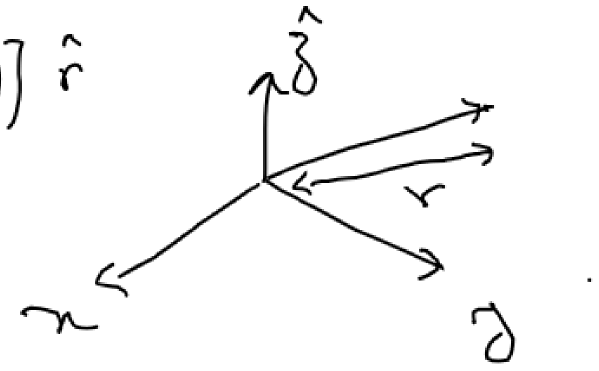
$$\vec{F}(\vec{r}) = [F_\rho(\rho)]\,\hat{\rho}$$

In fact, the force is three dimensional, not two dimensional. We are living in a 3 dimensional space, not a 2 dimensional plane. And the force should be written as

$$\vec{F}(\vec{r}) = [F_r(r)]\,\hat{r}$$



r is the distance from the origin

$$\hat{r} = \frac{\vec{r}}{r} \qquad r = |\vec{r}|$$

But it can be seen that as the force is radial, the motion will be restricted to a plane:

The angular momentum is defined as the cross product of the position and the linear momentum:

$$\underbrace{\vec{L}}_{\text{angular momentum}} = \underbrace{\vec{r}}_{\text{position}} \times \underbrace{\vec{p}}_{\text{linear momentum}}$$

linear momentum $= m\vec{v}$

mass

velocity

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

The first term vanishes:

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times (m\vec{v}) = 0$$

$\vec{v}$ is parallel to $(m\vec{v})$

The time derivative of the linear momentum is the force:

$$\vec{F} = \frac{d\vec{P}}{dt} = m\frac{d\vec{v}}{dt}$$ Newton's second law

$$\frac{d\vec{L}}{dt} = \underbrace{\vec{r} \times \vec{F}}$$ → the torque

For a cental force $\vec{F} \parallel \vec{r}$

$\vec{F} = (\text{something})\,\hat{r} \qquad \hat{r} \parallel \vec{r}$

so $\vec{r} \times \vec{F} = 0$, if the force is central.

So, if the force is central, the angular momentum is conserved. (It is a constant, its time derivative is zero.)

$$\vec{L} = \vec{r} \times \vec{p}$$

$\vec{L}$ is a constant (central force)

which means that both the direction and magnitude of $\vec{L}$ are constants.

The cross product of two vectors is normal to both vectors. So the angular momentum is normal to the position vector.

But the direction of the angular momentum is fixed (with a central force).
So the poistion vector is always normal to a fixed direction.

That means that the position vector is in a fixed plane: the plane which is normal to that direction (the direction of the angular momentum).

This is true for any central force, not necessarily gravity: With any central force, the motion is on a plane.

Now that we know that the motion is on a plane, we can choose the coordinates in a suitable way:
The z axis is chosen so that the plane of the motion is the x-y plane (the plane z=0),
and the origin is on the source of the force.

With these, the above choice: $z \neq 0$

$$\vec{r} = r\hat{r} = \rho\hat{\rho}$$

$$\vec{F}(\vec{r}) = [F_\rho(\rho)]\hat{\rho}$$

The constancy of the direction of the angular momentum was used to prove that the motion is restricted to a plane. What about the constancy of the amgnitude of the angular momentum?

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{r} = \rho\hat{\rho} \qquad \vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi}$$

$$\vec{L} = (\rho\hat{\rho}) \times [m(\dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi})] \qquad \hat{\rho} \times \hat{\rho} = 0$$

$$\vec{L} = m\rho^2\dot{\varphi}\hat{z} \qquad \hat{\rho} \times \hat{\varphi} = \hat{z}$$

So the angular momentum is parallel to the z axis, which means that the position vector is normal to the z axis. (This is not new, it was already known).

But the angular momentum is a constant. So its magnitude is also a constant:

$m\rho^2\dot{\varphi} = \text{constant}$

This $(m\rho^2\dot{\varphi})$ is not necessarily the magnitude of $\vec{L}$, because $(m\rho^2\dot{\varphi})$ can be negative.

The magnitude of $\vec{L}$ is $|m\rho^2\dot{\varphi}|$

But $\vec{L}$ is a constant

$\vec{L} = (m\rho^2\dot{\varphi})\hat{z}$ showing that $m\rho^2\dot{\varphi} = \text{constant}$.

So, $m\rho^2\dot{\varphi} = \text{constant}$, the meaning of this quantity?

Recall that

$$\frac{dS}{dt} = \frac{1}{2}\rho^2\dot{\varphi}$$ The surface area scanned by the position vector

$$m\rho^2\dot{\varphi} = (2m)\left(\frac{1}{2}\rho^2\dot{\varphi}\right)$$

$$m\rho^2\dot{\varphi} = \text{constant} \Longleftrightarrow \frac{dS}{dt} = \text{constant}$$

(vector)
The angular momentum is a constant
$\Longleftrightarrow$ The motion is restricted to a plane, and the time derivative of the surface area scanned by the position vector is a constant.

The first and the third Kepler laws

A simpler method:

Using Kepler's third law:

$$\frac{T^2}{a^3} = \text{Constant}$$ (for anything orbiting the sun

T: period

a: semi-major axis

A special case: A circular orbit:

$a = R \rightarrow$ the radius

$$\frac{T^2}{R^3} = \text{constant}$$

The period T can be found from the form of the force F:

For motion on a circle:

$\vec{r} = \rho\hat{\rho} \quad \rho = R$ (constant)

$$\vec{v} = \underset{\downarrow \atop =0}{\dot{\rho}}\,\hat{\rho} + \rho\dot{\varphi}\,\hat{\varphi} = R\dot{\varphi}\,\hat{\varphi}$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\,\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\,\hat{\varphi}$$

$$= -R\dot{\varphi}^2\,\hat{\rho} + R\ddot{\varphi}\,\hat{\varphi}$$

Newton's second law:

$$m\vec{a} = \vec{F}$$

$$m\left[-R\dot{\varphi}^2\,\hat{\rho} + R\ddot{\varphi}\,\hat{\varphi}\right) = [F(\rho)]\,\hat{\rho} = [F(R)]\,\hat{\rho}$$

$$mR\ddot{\varphi} = 0 \qquad -mR\dot{\varphi}^2 = F(R)$$

$$\dot{\varphi}^2 = -\frac{F(R)}{mR} \qquad F(R) < 0$$

The force shoud be attractive.

No circular orbit is possible with a repulsive force.

$\dot{\varphi}$ = constant

$$|\dot{\varphi}| = \sqrt{-\frac{F(R)}{mR}}$$

T : the period

As $\dot{\varphi} =$ constant,

$$\dot{\varphi}\,\Delta t = \Delta\varphi \qquad \Delta t = T \qquad \Delta\varphi = \begin{cases} 2\pi, & \dot{\varphi} > 0 \\ -2\pi, & \dot{\varphi} < 0 \end{cases}$$

$$T = \begin{cases} \frac{2\pi}{\dot{\varphi}}, & \dot{\varphi} > 0 \\ -\frac{2\pi}{\dot{\varphi}}, & \dot{\varphi} < 0 \end{cases} \longrightarrow T = \frac{2\pi}{|\dot{\varphi}|}$$

$$T^2 = \frac{4\pi^2}{\dot{\varphi}^2} \qquad \dot{\varphi}^2 = -\frac{F(R)}{mR}$$

$$\frac{4\pi^2}{T^2} = -\frac{F(R)}{mR}$$

Correct for any attractive central force with a cicular orbit.

Kepler's third law (for a circular orbit)

$$\frac{T^2}{R^3} = \text{constant}$$ (depending on anly the properties of the sun)

This constant (C) is independent of R or the properties of the planet (anything orbiting the sun)

$$\frac{T^2}{R^3} = C \qquad T^2 = CR^3$$

$$\frac{4\pi^2}{T^2} = -\frac{F(R)}{mR} = \frac{4\pi^2}{CR^3}$$

This is an expression for F(R).

$$F(R) = -\frac{4\pi m}{CR^2}$$ this C is not the speed of light

F is propotional to the iverse of the square of R

$F(R) \propto \frac{1}{R^2}$ inverse square law

$F(R)$ is proportional to $\frac{1}{R^2}$.

The proportionality constant $\left(-\frac{4\pi^2 m}{C}\right)$ is a function of the properties of the planet (orbiting thing) and the sun.

Which properties?

Recall that C is independent of the arbiting thing: It depends an only the properties of the sun

$$\frac{4\pi^2}{C} = \mathcal{G}_{sun}$$

$$F(r) = -\frac{m\,\mathcal{G}_{sun}}{r^2}$$

the mass of the arbiting thing

A remarkable property for this force (gravity)

$$\vec{F} = -\frac{m\,\mathcal{G}_{sun}}{r^2}\hat{r}$$

$$\vec{a} = \frac{1}{m}\vec{F} = -\frac{\mathcal{G}_{sun}}{r^2}\hat{r}$$

The acceleration $\vec{a}$ is indenpent of the mover:

The orbits (paths) of two different thing with the same initial conditions (position and velocity) are the same.

Free fall is universal

Motion under gravity (only) doesn't depend on the properties of the mover.

Motion under gravity is determined by only the initial conditions (and, of course, the source of gravity).

This is not the case for anything apart from gravity.

For example: electrostatice force

$\vec{F} = q\,\vec{E}(\vec{r})$

the electic field

a function of the position

(and, of course, the source).

the electric charge of the mover.

$\vec{a} = \frac{q}{m}\vec{E}(\vec{r})$ $\left(\frac{q}{m}\right)$ depends on the mover

mass of the move

$\left(\frac{q}{m}\right)$ is not the same for all movers.

For gravity: $\vec{F} = m\,\vec{g}(\vec{r})$

depending on the source and the position.

$\vec{a} = \frac{\vec{F}}{m} = \vec{g}(\vec{r})$ independent of the properties of the mover.

Returning to the sun, and the solar system.

$$\vec{F}(\vec{r}) = -\frac{m\mathcal{G}_{sun}}{r^2}\hat{r}$$

$\downarrow$

the gravitational force of the sun applied on the mover.

What is $\mathcal{G}_{sun}$?

Using Newton's third law

$$\vec{F}_{mover \to sun} = -\vec{F}_{sun \to mover} = +\frac{m \mathcal{G}_{sun}}{r^2}\hat{r}$$

Writing $\vec{F}_{mover \to sun}$ similar to $\vec{F}_{sun \to mover}$:

$$\vec{F}_{mover \to sun} = -\frac{m_{sun}\,\mathcal{G}_{mover}}{\tilde{r}^2}\hat{\tilde{r}}$$

$\vec{\tilde{r}}$ is the position of the sun relative to the mover

$\tilde{\vec{r}} = -\vec{r}$

$\vec{r}$: the position of the mover relation to the sun

$$\vec{F}_{mover \to sun} = -\frac{m_{sun} G m_{mover}}{r^2}(-\hat{r})$$

$\tilde{\hat{r}} = -\hat{r}$ $\quad \tilde{r} = r$

$$= \frac{m_{sun} G m_{mover}}{r^2}\hat{r} = \frac{m_{mover} G m_{sun}}{r^2}\hat{r}$$

Newton's 3rd law

$$m_{sun} g_{mover} = m_{mover} g_{sun}$$

For any two objects (1 and 2)

$$m_1 g_2 = m_2 g_1$$

independent of 2

$$\Rightarrow \frac{g_2}{m_2} = \frac{g_1}{m_1}$$

So if 1 is changed but 2 is not changed, the right-hand side shouldn't change

depends on only 2

And, as the right handside is independent of 2

The right-hand side should be a constant:

$$\frac{\mathcal{G}_1}{m_1} = G \rightarrow \text{a universal Constand}$$

$$\mathcal{G}_1 = G m_1 \qquad m_2 \mathcal{G}_1 = G m_1 m_2$$

$$\vec{F}_{1\to 2} = -\frac{G m_1 m_2}{r^2}\hat{r} \qquad \vec{r} = \vec{r}_2 - \vec{r}_1$$

the position of 2 relative to 1

The Newtanian gravitational force

$$F_{1\to 2} = -\frac{G m_1 m_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

$$\frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3}$$

Kepler's empirical laws result in the above form for the Newtanian gravitational force.