$$P(M=m) = \binom{n}{m} p^m q^{n-m}$$

$pn = \lambda \qquad p \to 0$

$$P(M=m) = \frac{e^{-\lambda} \lambda^m}{m!}$$

$\lambda \to \infty$

دوجمله‌ای $\xrightarrow[n \to \infty]{}$ گاوسی

دوجمله‌ای

پواسن

گاوسی

مثل اینها

$$P(M=m) = \binom{n}{m} p^m q^{n-m}$$

$$n \longrightarrow \infty$$

$$\ln(j!) = j\ln j - j + \ln\sqrt{2\pi j} + \cdots$$

$$P(M=m) = e^{f(m)}$$

$$f(m) = n\ln n - n + \ln\sqrt{2\pi n} - \left[m\ln m - m + \ln\sqrt{2\pi m}\right)$$
$$- \left[(n-m)\ln(n-m) - (n-m) + \ln\sqrt{2\pi(n-m)}\right]$$
$$+ m\ln p + (n-m)\ln q$$

$$f'(m) = -(\ln m + 1 - 1) + [\ln(n-m) + 1 - 1] + \cdots$$

کمتر از ۱

$\frac{1}{n-m}, \frac{1}{m}$

$$+ \ln p - \ln q = \ln \frac{n-m}{m} + \ln \frac{p}{q}$$

$$f'(m_0) = 0 \qquad \ln \frac{n-m_0}{m_0} + \ln \frac{p}{q} = 0$$

$$\frac{n-m_0}{m_0} = \frac{q}{p} \qquad m_0 = np \qquad n - m_0 = nq$$

$$f''(m) = -\frac{1}{m} - \frac{1}{n-m} \qquad f''(m_0) = -\frac{n}{m_0(n-m_0)}$$

$$f''(m_0) = -\frac{1}{npq}$$

$$f(m) = f(m_0) + \frac{f''(m_0)}{2}(m-m_0)^2 + \cdots$$

$$\overset{m_0 + (n-m_0)}{=} n\ln n - \cancel{n} + \ln\sqrt{2\pi n} - \left(m_0 \ln m_0 - \cancel{m_0} + \ln\sqrt{2\pi m_0}\right)$$

$$- \left[(n-m_0)\ln(n-m_0) - \cancel{(n-m_0)} + \ln\sqrt{2\pi(n-m_0)}\right]$$

$$+ m_0 \ln p + (n-m_0)\ln q - \frac{(m-m_0)^2}{2npq} + \cdots$$

$$= \overset{np}{m_0} \ln \overset{1/p}{\frac{n}{m_0}} + \overset{nq}{(n-m_0)} \ln \overset{1/q}{\frac{n}{n-m_0}}$$

$$- \ln\sqrt{2\pi m_0(n-m_0)/n} \quad + n(p\ln p + q\ln q)$$

$$-(m-m_0)^2/(2npq) + \cdots$$

$$f(m) = -\frac{(m-m_0)^2}{2npq} - \ln\sqrt{\frac{2\pi m_0(n-m_0)}{n}} + \cdots$$

$$= -\frac{(m-np)^2}{2npq} - \ln\sqrt{2\pi npq} + \cdots$$

$$P(M=m) = e^{f(m)} = \frac{e^{-\frac{(m-np)^2}{2npq}+\cdots}}{\sqrt{2\pi npq}}$$

$$\Delta X = \frac{\Delta M}{\sqrt{n}}$$

$$\frac{m-np}{\sqrt{n}} = x \qquad \Delta x = \frac{1}{\sqrt{n}}\Delta m \qquad \frac{M-np}{\sqrt{n}} = X$$

$$\rho_X(x)\,\Delta X = P(M=m)\,\Delta M$$

$$\rho_X(x) = \frac{P(M=m)\,\Delta M}{\Delta X} = \sqrt{n}\;P(M=m)$$

$$\rho_X(x) = \frac{e^{-\frac{x^2}{2pq_h}}}{\sqrt{2\pi p q_h}} \qquad n \to \infty$$

گشتاورهای توزیع گاوسی (یک متغیره)

$$f_X(x) = \frac{e^{-\frac{(x-a)^2}{2b^2}}}{\sqrt{2\pi b^2}} \qquad b>0$$

$$x-a=y$$

$$\langle X\rangle = \int_{-\infty}^{+\infty} dx \, \frac{e^{-\frac{(x-a)^2}{2b^2}}}{\sqrt{2\pi b^2}}\, x$$

زوج

صفر

$$= \int_{-\infty}^{+\infty} dy \, \frac{e^{-\frac{y^2}{2b^2}}}{\sqrt{2\pi b^2}} (y+a) = a\int_{-\infty}^{+\infty} dy \, \frac{e^{-\frac{y^2}{2b^2}}}{\sqrt{2\pi b^2}}$$

$$= a$$

$$\langle x \rangle = a$$

$$\mathrm{Var}(x) = \langle (x - \langle x \rangle)^2 \rangle = \langle (x-a)^2 \rangle$$

$$= \int_{-\infty}^{+\infty} dx \, \rho_x(x) \, (x-a)^2 = \int_{-\infty}^{+\infty} dy \, \frac{e^{-\frac{y^2}{2b^2}}}{\sqrt{2\pi b^2}} \, y^2$$

$$\int_{z_1}^{z_2} dz \, e^{-\alpha z^2} z^l = -\frac{d}{d\alpha} \int_{z_1}^{z_2} dz \, e^{-\alpha z^2} z^{l-2}$$

$$\mathrm{Var}(x) = \frac{1}{\sqrt{2\pi b^2}} \int_{-\infty}^{+\infty} dy \; e^{-\frac{y^2}{2b^2}} \; y^2$$

$$\int_{-\infty}^{+\infty} dy \; e^{-\alpha y^2} \; y^2 = -\frac{d}{d\alpha} \int_{-\infty}^{+\infty} dy \; e^{-\alpha y^2}$$

$$= -\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \qquad \alpha = \frac{1}{2b^2}$$

$$\mathrm{Var}(x) = b^2 \frac{1}{\sqrt{2\pi b^2}} \int_{-\infty}^{+\infty} dy \; e^{-\frac{y^2}{2b^2}} = b^2$$

$$\rho_X(x) = \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{(x-a)^2}{\sqrt{2\pi b^2}}}$$

$$\langle X \rangle = a \qquad \mathrm{Var}(X) = b^2$$

$$\sigma(X) = b$$

$$\rho_X(x) = \frac{1}{\sqrt{2\pi\sigma^2(X)}} e^{-\frac{(x-\langle X\rangle)^2}{2\sigma^2(X)}}$$

$$\sigma(X) \to 0 \qquad \rho_X(x) \to \delta(x - \langle X \rangle)$$

قضیه‌ی حد مرکزی:

متغیرهای تصادفی X_1 تا X_n

توزیع X_iها یکسان، و X_iها ناوابسته اند.

$$X = X_1 + \cdots + X_n$$

$n \to \infty$: توزیع احتمال X ← ?

X , Y متغیرهای تصادفی:

$$\rho_{X,Y} \longrightarrow \rho_{X+Y}$$

$$\rho_{f(X)}(s) = \langle \delta[f(X) - s] \rangle$$

$$\rho_{X+Y}(s) = \langle \delta(X + Y - s) \rangle$$

$$= \int dx\, dy\; \rho_{X,Y}(x, y)\, \delta(x + y - s)$$

$$= \int dx\; \rho_{X,Y}(x, s - x)$$

$$\rho_{f(x)}(s) \overset{?}{=} \langle \delta[f(x)-s] \rangle$$

$$f(x) = w \qquad \langle g(w) \rangle$$

$$\langle g(w) \rangle = \int d\omega \, \rho_w(\omega) \, g(\omega)$$

$$\langle g(w) \rangle = \int d\cancel{x} \, \rho_x(x) \, g[f(x)]$$

$$= 1 \int dx \, \rho_x(x) \, g[f(x)] = \int dx \, \rho_x(x) \, g[f(x)] \, 1$$

$$= \int dx \, ds \, \rho_x(x) \, g[f(x)] \, \delta[f(x)-s]$$

$$\langle g[f(x)]\rangle = \int dx\, ds\, \rho_x(x)\, g(s)\, \delta[f(x)-s]$$

$$= \int ds\, g(s) \int dx\, \rho_x(x)\, \delta[f(x)-s]$$

$$= \int ds\, g(s) \langle \delta[f(x)-s] \rangle$$

$f(x) = w \qquad \langle g(w)\rangle = \int dw\, \rho_w(w)\, g(w)$

$$\langle g(w)\rangle = \int dw\, \langle \delta[f(x)-w]\rangle\, g(w)$$

$\forall g: \quad \rho_w(w) = \delta[f(x)-w]$

$$\rho_{f(X)}(\omega) = \langle \delta[f(X) - \omega] \rangle$$

$$\rho_{X+Y}(s) = \int dx\, \rho_{X,Y}(x, s-x)$$

$$\rho_{X,Y}(x, y) = \int \frac{dk}{2\pi} \frac{dl}{2\pi} e^{i(kx+ly)} \tilde{\rho}_{X,Y}(k, l)$$

$\tilde{\rho}$: تبدیل فوریه ی ρ

$$\tilde{\rho}_{X,Y}(k, l) = \int dx\, dy\, \rho_{X,Y}(k, l)\, e^{-i(kx+ly)}$$

$$\rho_{X+Y}(s) = \int dx \frac{dk}{2\pi}\frac{dl}{2\pi}\tilde{\rho}_{X,Y}(k,l)\, e^{ikx+il(s-x)}$$

$$\int dx\, e^{iqx} = 2\pi\,\delta(q)$$

$$\rho_{X+Y}(s) = \int \frac{(dk)(dl)}{2\pi}\tilde{\rho}_{X,Y}(k,l)\, e^{ils}\,\delta(k-l)$$

$$= \int \frac{dk}{2\pi}\tilde{\rho}_{X,Y}(k,k)\, e^{iks}$$

$$= \int \frac{dk}{2\pi}\tilde{\rho}_{X+Y}(k)\, e^{iks} \qquad \tilde{\rho}_{X+Y}(k) = \tilde{\rho}_{X,Y}(k,k)$$

$$\tilde{\rho}_{X+Y}(k) = \tilde{\rho}_{X,Y}(k,k)$$

کلی: لازم نیست
X و Y ناوابسته باشند.

اگر X و Y ناوابسته باشند،

$$\rho_{X,Y}(x,y) = \rho_X(x)\,\rho_Y(y)$$

$$\tilde{\rho}_{X,Y}(k,l) = \int dx\,dy\,\rho_{X,Y}(x,y)\,e^{-i(kx+ly)}$$

$$= \int dx\,dy\,\rho_X(x)\,\rho_Y(y)\,e^{-i(kx+ly)}$$

$$\tilde{\rho}_{X,Y}(k,l) = \int dx\, \rho_X(x)\, e^{-ikx} \int dy\, \rho_Y(y)\, e^{-ily}$$

$$= \tilde{\rho}_X(k)\, \tilde{\rho}_Y(l)$$

X و Y نابسته ←

$$\tilde{\rho}_{X,Y}(k,l) = \tilde{\rho}_X(k)\, \tilde{\rho}_Y(l)$$

$$\tilde{\rho}_{X+Y}(k) = \tilde{\rho}_{X,Y}(k,k)$$ کلاً (بدون شرط نابستگی)

$$\tilde{\rho}_{X+Y}(k) = \tilde{\rho}_X(k)\, \tilde{\rho}_Y(k)$$ ← X, Y نابسته

X_1 تا X_n مستقل:

$$\tilde{\rho}_{X_1+\cdots+X_n}(k) = \tilde{\rho}_{X_1}(k)\cdots\tilde{\rho}_{X_n}(k)$$

$$\rho_{X_1+\cdots+X_n}(s) = \int \frac{dk}{2\pi}\, \tilde{\rho}_{X_1}(k)\cdots\tilde{\rho}_{X_n}(k)\, e^{iks}$$

$$\rho_X(x) \qquad \tilde{\rho}_X(k) = \int dx\, \underbrace{\rho_X(x)}_{\text{حقیقی و نامنفی}}\, \underbrace{e^{-ikx}}_{۱ \div ۱\ ۱}$$

$$|\tilde{\rho}_X(k)| = \left| \int dx\, \rho_X(x)\, e^{-ikx} \right|$$

$$\leq \int dx\, \left| \rho_X(x)\, e^{-ikx} \right| = \int dx\, \rho_X(x) \overbrace{\left| e^{-ikx} \right|}^{1}$$

$$= \underbrace{\int dx\, \rho_X(x)}_{=1}$$

$$|\tilde{\rho}_X(k)| \leq 1$$

$$\tilde{\rho}_X(0) = \int dx\, \rho_X(x) = 1$$

یعنی، $|\tilde{\rho}_X(k)|$ در $k=0$ ماکزیمم است.

작은 k

$$\tilde{\rho}_X(k) = \int dx\, \rho_X(x) \left[1 - ikx + \frac{(-ikx)^2}{2!} + \cdots \right]$$

$$= 1 - ik \int dx\, \rho_X(x) x - \frac{k^2}{2} \int dx\, \rho_X(x) x^2 + \cdots$$

$$= 1 \underbrace{- ik\langle X \rangle - \frac{k^2}{2} \langle X^2 \rangle}_{\varepsilon} + \cdots$$

$$\ln\left[\tilde{\rho}_X(k)\right] = -ik\langle X \rangle - \frac{k^2}{2}\langle X^2 \rangle - \frac{(-ik\langle X \rangle)^2}{2} + \cdots$$

$$= -ik\langle X \rangle - \frac{k^2}{2} \mathrm{Var}(X) + \cdots$$

$$\ln(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + \cdots$$

$$\tilde{\rho}_X(k) = e^{-ik\langle X\rangle - \frac{k^2}{2}\mathrm{Var}(X) + \cdots},$$

$$\rho_{X_1+\cdots+X_n}(s) = \int \frac{dk}{2\pi}\, \tilde{\rho}_{X_1}(k)\cdots\tilde{\rho}_{X_n}(k)\, e^{iks}$$

$$= \int \frac{dk}{2\pi}\, e^{-ik\sum_j \langle X_j\rangle - \frac{k^2}{2}\sum_j \mathrm{Var}(X_j) + \cdots}\, e^{iks}$$

[illegible] ... : $n \to \infty$

$$\rho_{\sum X_j}(s) = \int \frac{dk}{2\pi}\, e^{-\frac{k^2}{2}\sum_j \mathrm{Var}(X_j) + ik\left(s - \sum_j \langle X_j\rangle\right)}$$

$$\mathrm{Var}(X+Y)=\langle (X+Y-\langle X+Y\rangle)^2\rangle$$

$$=\langle (X+Y-\langle X\rangle-\langle Y\rangle)^2\rangle$$

$$=\underbrace{\langle (X-\langle X\rangle)^2}_{\mathrm{Var}(X)}+\underbrace{\langle (Y-\langle Y\rangle)^2\rangle}_{\mathrm{Var}(Y)}+2\langle (X-\langle X\rangle)(Y-\langle Y\rangle)\rangle$$

$\downarrow$ = [illegible] X, Y

$$\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)$$

$\langle (X-\langle X\rangle)\rangle\langle (Y-\langle Y\rangle)\rangle$

$\underset{=0}{\downarrow}$ $\underset{=0}{\downarrow}$

اگر X و Y نابسته باشند

(کافیست X و Y ناهمبسته باشند)

$$\rho_{\sum x_j}(s) = \int \frac{dk}{2\pi}\, e^{-\frac{\sum_j Var(x_j)}{2}\left[k - i\frac{s-\sum_j \langle x_j\rangle}{\sum_j Var(x_j)}\right]^2}$$

$$\times\ e^{-\frac{(s-\sum_j \langle x_j\rangle)^2}{2\sum_j Var(x_j)}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi}{\sum_j Var(x_j)}}\ e^{-\frac{(s-\sum_j\langle x_j\rangle)^2}{2\sum_j Var(x_j)}}$$

$$\sum_j x_j = X \qquad \rho_X(x) = \frac{1}{\sqrt{2\pi Var(X)}}\, e^{-\frac{(x-\langle X\rangle)^2}{2Var(X)}}$$

$$\sum_j \langle x_j\rangle = \langle X\rangle$$

$\sum_j Var(x_j) = Var(X) \longrightarrow$ x_j ها نابسته‌اند.

X_1, X_n تا بسته باشند، $n \to \infty$

$\left[\sum_j \mathrm{Var}(X_j)\right] \to \infty$ (از جمله آخر ترم‌ها کوچک باشند)

$X = \sum_j X_j$ توزیع X گاوسی می‌شود

قضیه‌ی حد مرکزی:

$$\langle X \rangle = \sum_j \langle X_j \rangle$$

$$\mathrm{Var}(X) = \sum_j \mathrm{Var}(X_j)$$