$$X \to \vec{X} : \quad x_1, \dots, x_n$$

$$\rho_X(x) = \mathcal{N} \, e^{-\frac{1}{2}\sum_{i,j}(x_i - a_i) M_{ij} (x_j - a_j)}$$

$$M_{ij} = M_{ji}$$

$$1 = \int d^n x \; \rho_X(x)$$

$$e^{-\frac{1}{2}\left[M_{11}(x-b)^2 + 2M_{12}(x-b)(y-c) + M_{22}(y-c)^2\right]}$$

$$\int_{-\infty}^{+\infty} dy \downarrow = \int dy \, e^{-\frac{1}{2}\left\{M_{11}(x-b)^2 + \left[M_{22}(y-c) + \frac{M_{12}(x-b)}{M_{22}}\right]^2 - \frac{(M_{12})^2 (x-b)^2}{M_{22}}\right\}} = \sqrt{\frac{2\pi}{M_{22}}} \; e^{-\frac{1}{2}\left[M_{11} - \frac{(M_{12})^2}{M_{22}}\right](x-c)^2}$$

$$\int dx\,dy\,\rho(x,y) = \sqrt{\frac{2\pi}{M_{22}}}\int dx\, e^{-\frac{1}{2}\left[M_{11}-\frac{(M_{12})^2}{M_{22}}\right](x-b)^2}$$

$$= \sqrt{\frac{2\pi}{M_{22}}}\sqrt{\frac{2\pi}{M_{11}-\frac{(M_{12})^2}{M_{22}}}} = \sqrt{\frac{(2\pi)^2}{\left[M_{11}M_{22}-(M_{12})^2\right]}}$$

$$= \sqrt{\frac{(2\pi)^2}{\det M}}$$

$$\rho_X(x) = \tilde{N}\, e^{-\frac{1}{2}\left[\sum_{ij} M_{ij} x_i x_j + 2\sum_i f_i x_i + E\right]}$$

$$= \tilde{N}\, e^{-\frac{1}{2}\left[x^t M x + (f^t x + x^t f) + E\right]}$$

$$f = -Ma \qquad a = -M^{-1} f$$

$$x^t f = -x^t M a \qquad f^t x = -a^t M^t x = -a^t M x$$

$$x^t M x + f^t x + x^t f = (x^t - a^t) M (x-a) - a^t M a$$

$$\rho_X(x) = \tilde{N}\, e^{-\frac{1}{2}\left[(x^t - a^t) M (x-a) + E - a^t M a\right]}$$

$$= N\, e^{-\frac{1}{2}\left[(x^t - a^t) M (x-a)\right]}$$

$M^t = M$

M حقیقی و متقارن

$M = U D U^t$

D قطری

$U^t = U^{-1}$

↓

اعضای روی قطر ویژه مقدار

ستون‌های U ویژه‌بردارهای یکه - متعامد

$$(x-a)^t M (x-a) = (x-a)^t U D U^t (x-a)$$

$(x-a) = U y$

$$= y^t D y$$

$$\int d^n x \, \rho_X(x) = \int d^n x \, \mathcal{N} \, e^{-\frac{1}{2} y^t D y}$$

$$y^t D y = \sum_i \mu_i (y_i)^2$$

μ_i ها اعضای روی قطر D اند.

$$x - a = U y \qquad d^n x = |\det(U)| \, d^n y$$

$$U^t = U^{-1} \qquad [\det(U)]^2 = 1$$

$$d^n x = d^n y$$

$$\int d^n x \, \rho_X(x) = \mathcal{N} \int d^n y \; e^{-\frac{1}{2}\sum_j \mu_j (y_j)^2}$$

$$= \mathcal{N} \prod_j \int dy_j \, e^{-\frac{1}{2}\mu_j (y_j)^2}$$

$$= \mathcal{N} \prod_j \sqrt{\frac{2\pi}{\mu_j}} \qquad \mu_j > 0$$

$$= \mathcal{N} \sqrt{\frac{(2\pi)^n}{\prod_j \mu_j}} \qquad M = U D U^t$$

$$\det(M) = [\det(U)]^2 \times \det(D)$$

$$\det(M) = \det(D) = \prod_j \mu_j$$

$$\int d^n x \; e^{-\frac{1}{2}(x-a)^t M (x-a)} = \sqrt{\frac{(2\pi)^n}{\det(M)}}$$

$$\int dx \; e^{-\frac{1}{2}\frac{(x-a)^2}{b^2}} = \sqrt{2\pi b^2}$$

$$(b^2)^{-1} \longrightarrow M \qquad \det\left[(b^2)^{-1}\right] = \frac{1}{b^2}$$

$$\mathcal{N}\sqrt{\frac{(2\pi)^n}{\det(M)}} = 1 \qquad \mathcal{N} = \sqrt{\frac{\det(M)}{(2\pi)^n}}$$

و چون مقدارهای M همان μ_i ها یعنی که مثبت اند

(پس صفر نیستند). پس M وارون‌پذیر است.

$$M^{-1} = C$$

$$\rho_X(x) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} e^{-\frac{1}{2}(x-a)^t C^{-1}(x-a)}$$

$$= \frac{1}{\sqrt{\det(2\pi C)}} e^{-\frac{1}{2}(x-a)^t C^{-1}(x-a)}$$

$$\langle X_i \rangle \qquad \langle (X_i - \langle X_i \rangle)(X_j - \langle X_j \rangle) \rangle$$

$$= \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle$$

$$X - a = U Y \qquad M = U D U^t = C^{-1}$$

$$\langle Y_i \rangle \, , \qquad \langle Y_i Y_j \rangle - \langle Y_i \rangle\langle Y_j \rangle$$

$$\langle Y_i \rangle = \int d^n y \, \rho_Y(y) \, y_i$$

$$\rho_Y(y) = \underbrace{\left| \det\left(\frac{\partial x}{\partial y}\right) \right|}_{=1} \rho_X(x)$$

$$x - a = U y$$

$$\frac{\partial x}{\partial y} = U$$

$$\rho_Y(y) = \rho_X(\overbrace{a + U y}^{x})$$

$$\langle Y_i \rangle = \frac{1}{\sqrt{\det(2\pi C)}} \int d^n y \, e^{-\frac{1}{2} \sum_j M_j (y_j)^2} y_i$$

$$\int dy_i \; e^{-\frac{1}{2}\mu_i (y_i)^2} \, y_i = 0$$

پس حاصل ضرب انتگرال ها هم
صفر میشود

$$\langle Y_i \rangle = 0 \qquad X - a = U\,Y$$

$$\langle X \rangle - a = U \langle Y \rangle = 0$$

$$\langle X \rangle = a \qquad \langle X_i \rangle = a_i$$

$\langle Y_i \rangle = 0 \qquad \langle Y_i Y_j \rangle$

$$\langle Y_1 Y_2 \rangle = \mathcal{N} \overbrace{\left[\int dy_1\, e^{-\frac{\mu_1}{2}(y_1)^2} y_1\right]}^{=0}$$

$$\times \underbrace{\left[\int dy_2\, e^{-\frac{\mu_2}{2}(y_2)^2}\right]}_{=0} \prod_{j\geq 3} \int dy_j\, e^{-\frac{\mu_j (y_j)^2}{2}}$$

$\langle Y_1 Y_2 \rangle = 0 \qquad \langle Y_i Y_j \rangle = 0, \quad j \neq i$

$\forall i$

$i=j$ $\quad \langle (Y_1)^2 \rangle = \mathcal{N} \int dy_1\, e^{-\frac{\mu_1}{2}(y_1)^2} (y_1)^2$

$$\times \prod_{j=2}^{n} \int dy_j\, e^{-\frac{\mu_j}{2}(y_j)^2} \quad {}^{\prime\prime}1$$

$$= \mathcal{N} \frac{1}{2\frac{\mu_1}{2}} \int dy_1\, e^{-\frac{\mu_1}{2}y_1^2} \prod_{j=2}^{n} \int dy_j\, e^{-\frac{\mu_j}{2}(y_j)^2}$$

$$= \frac{1}{\mu_1}$$

$$\int dy\, e^{-\alpha y^2} y^2 = \frac{1}{2\alpha} \int dy\, e^{-\alpha y^2}$$

$$\langle (Y_i)^2 \rangle = \frac{1}{\mu_i}$$ یعنی

$$\langle Y_i Y_j \rangle = \frac{1}{\mu_i}\delta_{ij} = (D^{-1})_{ij}$$

$$\langle Y_i \rangle \langle Y_j \rangle = 0 \qquad D_{ij} = \mu_i \delta_{ij}$$

$$\langle Y_i Y_j \rangle - \langle Y_i \rangle \langle Y_j \rangle = (D^{-1})_{ij}$$

$$\langle (X-a)_i (X-a)_j \rangle - \langle (X-a)_i \rangle \langle (X-a)_j \rangle$$

$$X - a = U Y$$

$$= \sum_{k,l} U_{ik} U_{jl} \left(\langle Y_k Y_l \rangle - \langle Y_k \rangle \langle Y_l \rangle \right)$$

$$= \sum_{k,l} U_{ik} U_{jl} (D^{-1})_{kl} = (U D^{-1} U^{-1})_{ij}$$

$$(U^t)_{lj} = (U^{-1})_{lj} \qquad = (U D^{-1} U^t)_{ij}$$

$$a_i = \langle x_i \rangle \qquad \langle (x_i - \langle x_i \rangle) \rangle = 0$$

$$\langle (x_i - a_i)(x_j - a_j) \rangle - \underbrace{\langle (x_i - a_i) \rangle}\langle (x_j - a_j) \rangle$$

$$= (U D^{-1} U^{-1})_{ij} = \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle$$

$$C(x_i, x_j) = (U D^{-1} U^{-1})_{ij} = (U D^{-1} U^t)_{ij}$$

$$M^{-1} = C \qquad M = U D U^{-1} = U D U^t$$

$$M^{-1} = (U^{-1})^{-1} D^{-1} U^{-1} = U D^{-1} U^{-1} = U D^{-1} U^t$$

$$C(x_i, x_j) = C_{ij} \rightarrow$$ همورداری = (همبستگی)

$$\rho_x(x) = \frac{1}{\sqrt{\det(2\pi C)}}\, e^{-\frac{1}{2}(x-\langle x\rangle)^t C^{-1} (x-\langle x\rangle)}$$

برای توزیع گاوسی ناهمبسته بودن : C قطری

$$\rho_x(x) = \mathcal{N}\, e^{-\frac{1}{2}\sum_i (C^{-1})_{ii}(x_i - \langle x_i\rangle)^2}$$

اگر C قطری باشد)

$$= \prod_i \rho_{x_i}(x_i)$$

$$\rho_{X_i}(x_i) = \frac{1}{\sqrt{2\pi (C^{-1})_{ii}}}\, e^{-\frac{1}{2}(C^{-1})_{ii}(x_i - \langle x_i \rangle)^2}$$

$$(C^{-1})_{ii} = (C_{ii})^{-1}$$

C قطری

یعنی X_1 تا X_n نابسته

$$\rho_X(x) = \prod_i \rho_{X_i}(x_i)$$

$\boxed{\text{حالت کلی: نابستگی} \Rightarrow \text{ناهمبستگی}}$

نابستگی $\Longleftrightarrow$ ناهمبستگی گاوسی